Indian J Phys **80** (11), 1075-1085 (2006)

Quantization of gravitons, gravity and sound waves†

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Abstract . We discuss some recent develoments in quantization of asymptotically $AdS_5 \times S^5$ geometries, with special emphasis on configurations preserving ≥ 4 supersymmetries We show that the spectrum on the boundary theory is reproduced by quantization of giant gravitons in AdS We discuss a novel bosonization which plays an important role in this equivalence By a simple generalization of this technique, we arrive at an exact solution of Tomonaga's problem dating back to 1950, which is that of bosonization of fermions one-dimensional metal

Keywords ADS/CFT duality, Bosonization, BPS constraints

PACS Nos 11 10 -z, 11 25 Tq, 11 25 Uv

Plan of Article

1 **Prologue**
2. **Introduction**
3 **Half-BPS geometries and fluctuations**
3 1 Spectrum of excitations in the bulk around $AdS_5 \times S^5$
3 1 1. BPS constraints and reduced phase space
3 1.2. Quantization · Spectrum of giants/dual giants
3 2 Half-BPS geometries
3.2 1 Examples of LLM geometries
3 3 Quantization of LLM gravity and AdS/CFT
3 4 Boundary theory
3 4.1. Interpretation of $u(x)$
3 4.2. Bosonization
3 4 3. Bosonization of the half-BPS system
3 5 Explicit counting of states and finite N
3 5.1. Finite N
3 6 Boundary description of singular geometries
1/8 BPS systems
4 1 Giant and dual giant gravitons
4 2 Quantization and comparison with the boundary theory
5. Less supersymmetries
5 1. Sasaki-Einstein geometries
5 2. $N_{susy} < 2$
6. Solution of Tomonaga's problem

1. Prologue

It is a great honour for me to dedicate this article to the memory of Prof Amal Kumar Raychaudhuri I am fortunate to be one of his students I am also fortunate to work in an area which gives me an opportunity to appreciate the power of his celebrated equation AKR, however, stands for much more than his equation, especially to his students He inspired a desire to understand things in their entirety, to bring out the lyrical beauty of physics, to use mathematics with the ease of a great conductor, and, in a nutshell, to emulate him as a teacher. Only later on, when I started doing my PhD, did I realise how hard it was to emulate his achievements as a researcher It is by now well-known how he arrived at his equation in complete isolation against all odds and fighting tremendous adversity As in the case of many other famous results, the full

† Dedicated to the memory of Proffessor Amal Kumar Raychaudhuri

import of Raychaudhuri equation has not perhaps been realized even now The most important recent discovery in string theory [1], that the world (at least if it is asymptotically anti de Sitter) appears to have a holographic description, may possibly be related to peculiar focussing properties of gravity encapsulated in Raychaudhuri equation (see, *e g* [2], Ch IV) It would be a fundamental advance to understand, in terms of known properties of field theories in flat space, such peculiar properties of gravity The following article summarizes some modest recent efforts in our understanding of the gauge-gravity duality in a supersymmetric context

2. Introduction

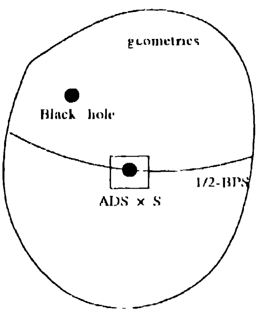

Figure 1. Asymptotically $AdS_5 \times S^5$ geometries

In recent years, there has been significant progress in quantizing geometries in asymptotically $AdS_5 \times S^5$ spaces, both directly as well as using *AdS/CFT* duality using $N = 4$ YM In this note we will describe some supersymmetric subsets of asymptotically $AdS_5 \times S^5$ geometries and low energy fluctuations around them and make some remarks about some nonsupersymmetric cases For a partial list of papers relevant to the supersymmetric configurations, see [3–24]

The first system that we will describe corresponds to half-BPS configurations which preserve 16 supersymmetries The bulk description of the system can be (a) in terms of half-BPS fluctuations such as gravitons [25] or giant gravitons [26–28] or (b) in terms of deformed geometries [3] which essentially back-reactions of such fluctuations when large The corresponding gauge theory description at the boundary is in terms of polynomials of traces of a single holomorphic scalar Z, which reduces to a theory of free fermions We will show an explicit correspondence among the quantum states in the bulk descriptions (a) and (b) and in the gauge theory description. An essential tool in this will be bosonization of these (non-relativistic) fermions in one-dimension where the bosons get related to the giant gravitons in the bulk [15–17] The correspondence between boundary states and giant gravitons can be extended to other systems [18–24] with number of supersymmetries down to 1/8-th (four) and also to $AdS_5 \times S^5$ spaces where the S^5 is replaced by 5-dimensional Sasaki-Einstein manifolds [30,29]

Curiously, the attempt to understand the above systems in gravity using bosonization leads to the solution of an apparently unrelated and unsolved problem that was first posed in condensed matter physics [31] In 1950, Tomonaga introduced the classic problem of finding a bosonic description of density waves in a metal. He considered a one-dimensional metal and represented it by N nonrelativistic fermions subject to mutual Coulomb interactions Although Tomonaga himself made significant progress (followed by others [32]) in achieving an approximate bosonization of fermions near the Fermi level and a large number of simpler versions including exact bosonization of relativistic fermions has been solved, the original Tomonaga problem of exact bosonization of N nonrelativistic fermions defied a solution In our study of half-BPS fluctuations around $AdS_5 \times S^5$ (giant gravitons), we found that the clue to this exact bosonization is provided by AdS/CFT duality where the fermions appeared in the boundary YM theory and the bosons were roughly the giant gravitons! In this note we will describe the exact bosonization of N nonrelativistic fermions in one dimension and mention its application to various problems, including the original problem of Tomonaga and that of giant gravitons

3. Half-BPS geometries and fluctuations

3.1 *Spectrum of excitations in the bulk around* $AdS_5 \times S^5$:

There are three different half-BPS low energy fluctuations around $AdS_5 \times S^5$ which have been studied extensively (1) gravitons [25], (2) giant gravitons [26] and (3) dual giant gravitons Myers, dgg-2. To understand these let us write the metric of $AdS_5 \times S^5$ in global coordinates

$$ds^2 = l^2(ds^2_{S^5} + ds^2_{AdS_5})$$

$$ds^2_{AdS_5} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho (d\Omega)^2$$

$$ds^2_{S_5} = -\cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta (d\tilde{\Omega})^2 \tag{3.1}$$

The coordinate ranges are $t \in (-\infty, \infty), \rho \in (0, \infty)$, $\theta \in (0, \pi/2), \phi \in (0, 2\pi)$

A giant graviton [26] is a D3-brane that sits at the centre of the AdS_5 ($\rho = 0$) and wraps the 3-sphere '$\tilde{S}^3$' inside S^5. The coordinates θ, ϕ are free to move. However, for BPS motions, it turns out that $\theta = 0$, $\dot{\phi} = 1$, that is, the giant graviton moves at the speed of light along a constant latitude θ. The hamiltonian and angular momentum are given by

$$H - p_\phi = \sin^2\theta = n/l, n = 0, 1, 2, \ldots, N-1 \tag{3.2}$$

where the last expression represents quantization (see Section 3.1.1). l represents the radius of curvature of the AdS (and 5-sphere) (see eq. (3.1)).

A dual giant graviton [27,28] is a D3-brane that wraps the 3-sphere 'S^3' part of AdS_5 and moves along a maximal circle of S^5, e.g. $\theta = 0$. For BPS motion, the AdS-size ρ is fixed and $\dot{\phi} = 1$, that is, the dual giant graviton moves at the speed of light along a maximal circle of S^5 at some fixed AdS radius ρ. In this case the hamiltonian and angular momentum are given by

$$H - p_\phi = \sinh^2\rho = n/l,\ n = 0, 1, 2, \ldots, \infty \tag{3.3}$$

where the last expression represents quantization (see Section 3.1.1).

A half-BPS graviton [25] is a Kaluza-Klein particle (a mixture of metric and RR fluctuations) rotating at the speed of light in the ϕ direction. Again the quantized hamiltonian and angular momentum are given by

$$H - p_\phi = n/l,\ n = 0, 1, 2, \ldots, \infty \tag{3.4}$$

The wavefunctions of these gravitons are given by $F_n(\rho,\theta)\exp[in\phi - int]$, where the F_n are localized functions which roughly behave as $(\cos\theta/\cosh\rho)^{|n|}$.

Comments

(i) The D-brane configurations have the same energy as the elementary gravitons. The reason is that the classical energy $\propto N \sim 1/g$ cancels due to supersymmetry and (3.2), (3.3) represent fluctuations which are $O(1)$ in N counting.

(ii) Giant gravitons see the stringy exclusion principle. Since the maximum size of the orbit is given by $\theta = 0$ (equator), in (3.2) the energy lH of a giant graviton cannot exceed N. There is a variant of this exclusion which operates for dual giant gravitons. Although the energy of individual dual giants is unbounded, their total ***number*** is bounded by N. To see this, begin at $\rho = \infty$ where the flux is N units; as we cross each dual giant, the flux decreases by 1. If we have N dual giants, we have zero flux inside and the space is flat which cannot support any more dual giants. This argument is not entirely rigorous since the dual giants are treated in the probe approximation in (3.3) whereas the above argument necessarily invokes back reaction on the geometry. We will, however, find two independent pieces of evidence for this argument below when we discuss (a) the back-reacted (LLM) geometries and (b) correspondence with the boundary theory.

(iii) The gravitons, (3.4), do not see the stringy exclusion principle. However, we will see that description of perturbative gravity in terms of gravitons itself breaks down for energies that scale faster than $\sqrt{N}$.

3.1.1. BPS constraints and reduced phase space :

We sketch the derivation of (3.2) and (3.3). For the giant graviton described above, the phase space, to begin with, is 4D, with coordinates $\theta, \phi, p_\theta, p_\phi$. The half-BPS condition imposes two second-class constraints, namely $p_\theta = 0$ and $p_\phi = \sin^2\theta$. We treat these by Dirac's method, leading to a 2D reduced phase with coordinates θ, ϕ which have a Dirac bracket

$$\left\{\sin^2\theta, \phi\right\}_{DB} = 1/N \tag{3.5}$$

This DB corresponds to the symplectic form of a hemisphere (note that $\theta \in [0,\pi/2)$. We can map this space to a unit disc in R^2 using $(x_1,x_2) = (\sin\theta\cos\phi, \sin\theta\sin\phi)$, with the canonical symplectic form

$$\omega = dx_1 \wedge dx_2 \tag{3.6}$$

The hamiltonian becomes that of a simple harmonic oscillator

$$H = \frac{1}{N}\left(x_1^2 + x_2^2\right) \tag{3.7}$$

For the dual giant, we begin with the 4D phase space ρ, ϕ, p_ρ, p_ϕ. Under the SUSY constraints $p_\rho = 0$, $p_\phi = \sinh^2\rho$, this gets reduced to a 2D phase space with Dirac bracket

$$\left\{\sinh^2\rho, \phi\right\}_{DB} = 1/N \tag{3.8}$$

Using $(x_1, x_2) = (\sinh\rho\cos\phi, \sinh\rho\sin\phi)$, we can map the system to R^2 with (3.6) as the symplectic form.

3.1.2 Quantization · Spectrum of giants/dual giants

Dual giants .

We found that the phase space of a single dual giant graviton is $C = R^2$, with the symplectic Kahler form

$$\omega = i dz \wedge d\bar{z} \tag{3.9}$$

Such a system can be quantized by the method of geometric quantization [33,34] using the Kähler structure. The result is that the wavefunctions are given by

$$\psi_n(z) = \psi_n^{hol}(z)\exp[-z\bar{z}/2],\ \psi_n^{hol}(z) = z^n,$$

$$(\psi, \psi) = \int d^2 z \psi(z) \psi(\bar{z}) \tag{3.10}$$

The harmonic oscillator hamiltonian (3.7) gets quantized as

$$H = z\frac{\partial}{\partial z} \tag{3.1}$$

which reproduces the spectrum (3.3) where $n = 0, 1, \ldots, \infty$

Giants

The phase space is now the Disc. The geometric quantization method now yields the finite spectrum (3.2)

Multiple giants/dual giants

Since dual giants are bosons, multi-particle wavefunctions have the form

$$\psi^{hol}_{s_1, s_2, \ldots, s_k}(z_1, \ldots, z_k) = \frac{1}{\sqrt{k!}} \sum_{\pi} \prod_i z_i^{s_{\pi}(i)}$$

$$\text{with energy eigenvalue } H = \sum_i s_i \tag{3.12}$$

where, following Comment (2.) in Section 3.1, we have the number of particles to be $k < N$.[1]

The above discussion can be summarized in terms of the second-quantized boson actions

$$S_g = \int dt \sum_{j=0}^{N} \left[\varphi_j^\dagger(-i\partial_t \varphi_j) + j\varphi_j^\dagger \varphi_j \right]$$

$$S_{dg} = \int dt \sum_{n=0}^{\infty} \left[\phi_j^\dagger(-i\partial_t \phi_n) + \left(n + \frac{1}{2} - \mu_B \right) \phi_n^\dagger \phi_j \right] \tag{3.13}$$

The first action corresponds to an arbitrary number of bosons (giant gravitons) in a harmonic oscillator truncated to the first N levels. The second action corresponds to N bosons (dual giants) in the standard harmonic oscillator potential (here we count dual giants as in footnote 1). The bosonic phase space densities, alluded to in footnote 2, can be constructed in much the same way as $u(x,p)$ is constructed from the second quantized fermion field in Sec. 3.4.1. Again there is an obvious correspondence between the bosonic oscillators introduced above and those in the boundary theory (Sec. 3.4).

[1] We can equivalently take the number of particles to be always N, provided we put $k' = N - k$ particles in the zero energy state, given by the constant wavefunction 1. The wavefunction (3.15) remains unchanged other than the fact that it becomes now a function of N z's.

$$\varphi_n \leftrightarrow a_n, \phi_n \leftrightarrow b_n \tag{3.14}$$

The fact that the bosonic action (3.13) is first order owes its origin to the BPS conditions.

In the above notation, the multiple dual giant states (3.12) can be written as

$$|\Psi\rangle_{dgg} = \prod_i^N \phi_{s_i}^\dagger |0\rangle, \phi_n|0\rangle = 0 \le s_N \le \ldots \le s_0$$

$$H = \sum_{i=1} s_i \tag{3.}$$

There is an obvious correspondence between the wavefunction and the boundary state (3.44)

The multiple-giant graviton wavefunction can similarly be written down from the previous analysis

$$|\Psi\rangle_{gg} = \prod_{i=1}^{N} (\varphi_i^\dagger)^{r_i} |0\rangle, \varphi_i|0\rangle = 0, r_i \ge 0$$

$$H = \sum_{i=1}^{N} i r_i \tag{3.}$$

again in an obvious correspondence with (3.40) in the boundary theory

See Section 3.5 for a detailed account of the matching between bulk and boundary

3.2 *Half-BPS geometries*

The spectra of particles described above are dealt with in the probe approximation, ignoring their back reaction. How does the back reaction change the geometry? In the half-BPS sector of asymptotically $AdS_5 \times S^5$ spaces, the complete set of solutions has been found by LLM. These geometries, like $AdS_5 \times S^5$, admit $O(4) \times O(4)$ isometries and contain two 3-spheres (the remaining coordinates are called t, y, $\vec{x} = (x_1, x_2)$). The geometry is static and depends on y, x_1, x_2. Explicitly, the metric is given by

$$ds^2 = -\sqrt{\frac{y}{u(1-u)}}(dt + V_i[u]dx_i)^2$$

$$+ \sqrt{\frac{u(1-u)}{y}}(dy^2 + dx_i dx_i)$$

$$+ y\sqrt{\frac{1-u}{u}}(d\Omega_3)^2 + y\sqrt{\frac{u}{1-u}}(d\tilde{\Omega}_3)^2 \tag{3.17}$$

where $u = u(y, x_1, x_2)$ is entirely specified by its value on the $y = 0$ plane

$$u(x_1, x_2, y) = \frac{y^2}{\pi}\int d^2x' \frac{u(x',0)}{((x-x')^2+y^2)^2} \quad (3.18)$$

The functions V_i too are determined by the function $u(x,0)$ [...] and so are the components of the RR five-form which we do not write explicitly here.

The value of the asymptotic flux for all these geometries [...]. This implies

$$\int \frac{d^2x}{2\pi\hbar} u(x) = N, \hbar = 2\pi l_p^4 \quad (3.19)$$

[...] as LLM argued, regularity of the geometries implies

$$u(x)^2 = u(x) \quad (3.20)$$

3.2 Examples of LLM geometries :

Example 1 The 'vacuum' :

$$u_{vacuum}(x) = \theta\left(r_0^2 - x_1^2 - x_2^2\right), r_0 = l^2 \quad (3.21)$$

represents $AdS_5 \times S^5$.

Example 2 Giant gravitons LLM suggested that 'holes' in the filled disc represent (geometry created by) giant gravitons and 'particles' (outside the disc) represent dual giants. Let us parametrize an a solution u of (3.20), (3.19) [...]

$$u(x) = u_{\text{filled disc}} - \sum_i \chi_{i\text{ th hole}}(x) + \sum_j \chi_{j\text{ th particle}}(x) \quad (3.22)$$

where $\chi_I(x) = 1$ if $x \in R$ and zero otherwise. We choose the 'holes' and 'particles' to be of minimum area $2\pi\hbar$ consistent with flux quantization. This, then corresponds to a geometry created by) giant gravitons at the location of the 'holes' and dual giants at the location of the 'particles'. For sufficiently generic configurations, far from $AdS \times S^5$ the notion of giants and dual giants are appropriately generalized [6].

Comments

(i) Duality between giants and dual giants. It's clear that the parametrization (3.22) is non-unique, *e.g.* a bunch of N circular rings (each of area $2\pi\hbar$) with radii $> r_0$ can be regarded in terms of holes or particles[2].

(ii) Stringy exclusion principle. The uv cut-off for giant gravitons follows from the fact that the deepest hole is at the origin whose energy is N (in $1/l$ units). If we consider the configuration mentioned in the above paragraph and regard each circular ring as a 'particle' (*i.e.* as a dual giant), clearly we cannot have more than N of them because of (3.19) (we can have less than N of them, *e.g.* if we have a disc of area m at the centre).

Example 3 Gravitons. If we assume a parametrization [7,8]

$$u(x) = \theta(f(\phi) - r^2), \quad (3.23)$$

then the Fourier modes $c_n, c_n^\dagger$ of $\delta f(\phi)$ correspond to the graviton fluctuations around the LLM geometry f. The wavefunctions are given by

$$|\Psi\rangle_k = \prod_n (c_n^\dagger)^{r_n}|0\rangle, n|0\rangle = 0, r_n \geq 0$$

$$H = \sum n r_n \quad (3.24)$$

Comparing with (3.15) and (3.16) we see that N does not figure in the multi-graviton wavefunction, in other words, there is no stringy exclusion principle operative for gravitons. Indeed, gravitons are not well-defined at energies $E > \sqrt{N}$ (see Section 3.4.3). As a result the the unbounded range of the single-particle energy n in the above equation is only valid in the limit $N = \infty$. Surprisingly, giant gravitons and dual giant gravitons make sense at finite N.

3.3 Quantization of LLM gravity and AdS/CFT

It is easy to see that $u(x)$ can be used to parametrize the space of the half-BPS vacua, namely the space of all functions on the plane subject to the constraints (3.20), (3.19). As shown in [35], for any given N, this space can be viewed as a coadjoint orbit of group w_∞ which is the group of area-preserving diffeomorphisms acting on the plane. The obvious way to quantize such a space is to Kirillov's quantization method, which leads to the following action

$$S_{\Lambda_w} = \int \frac{dx_1 dx_2}{2\pi\hbar}\hbar \int_{\hat\Sigma} dt\, ds\, u(x,t,s)\{\partial_t u, \partial_s u\}_{PB} - \int_\Sigma dt \tilde H$$

$$\tilde H = \int \frac{dx_1 dx_2}{2\pi\hbar} u(x,t) \frac{x_1^2 + x_2^2}{2\hbar} \quad (3.25)$$

In the above, the function $u(x, t)$ represents promotion of the collective coordinate $u(x)$ to a function of time (*cf.* for the 'tanh' kink in 1 + 1 dimensional scalar field theory centred on x_0, we promote $x_0 \to x_0(t)$). Σ represents a

[2] When we create a hole in the disc, we put the fermi fluid at the outer boundary expanding the disc a bit. Two contiguous circular holes correspond to two giant gravitons on top of each other, as a result the distribution u_g of giant gravitons is somewhat different from a distribution of holes in a Fermi fluid. The former, which can be derived from the latter by means of a certain transform, looks like an uneven pancake of radius r_0. The latter representation in terms of u looks rather like a flat pancake, with holes, and of a given total area (see Section [...]) and is the approximate fermionic representation.

(Feynman) path; the additional variable s, parametrizing the extension Σ (whose s-boundary is Σ), is introduced to write the symplectic form in a gauge-invariant way The measure of the functional integral, which we do not write here explicitly, incorporates the two constraints (3 20), (3 19)

Examples . The above action checks out with the various examples mentioned in the previous section For example, (3 21) clearly corresponds to the lowest energy state On the configuration (3 22) with non-overlapping holes and particles, the action (3 25) agrees with the combined DBI + CS action for the giants and dual giants The action also correctly reproduces the quantization of graviton fluctuations around $AdS_5 \times S^5$. The latter has been independently obtained using CWZ quantization in [8] where gravitons around arbitrary LLM geometry has also been discussed

It is useful to mention that (3 25) is equivalent to the semiclassical limit of a second quantized fermion action

$$S = \int dt\, dx\left(-i\psi^\dagger(x)\partial_t\psi(x) - x^2\psi^\dagger(x)\psi(x) - \mu\psi^\dagger(x)\psi(x)\right)$$

$$= \int dt \sum_{n=0}^{\infty}\left[\psi_n^\dagger(-i\partial_t\psi_n) + \left(n + \frac{1}{2} - u\right)\psi_n^\dagger\psi_n\right] \qquad (3\ 26)$$

where μ is the chemical potential enforcing the total number of fermions to be N. This is precisely the fermion action that describes the boundary theory, as we will in the next section This, therefore, constitutes a proof of AdS/CFT in the context of LLM geometries

3.4 Boundary theory :

The half-BPS sector of $N = 4$ SYM is described by operators of the form

$$O = \prod_i \mathrm{Tr}Z^{n_i} \equiv \prod_i O_{n_i} \qquad (3\ 27)$$

The complex matrix Z can be parametrized as

$$Z = UTU^\dagger \qquad (3\ 28)$$

where U is a unitary matrix and T is complex, upper triangular ($T_{ij} = 0$, $i > j$), with $T_{ii} = z_i$ being the eigenvalues of Z The choice of U is non-unique, but one can choose the gauge $d\Omega_{ii} = 0$, $d\Omega_{ij} \equiv -i[U^\dagger dU]_{ij}$ to fix the ambiguity The measure of integration over the complex matrix Z becomes

$$\prod dZ_{ij}d\bar{Z}_{ij} = \prod d\Omega_{ij}d\bar{\Omega}_{ij}\prod dT_{ij}d\bar{T}_{ij}\prod dz_i dz_i\,|\Delta(z)|^2$$

$$\Delta(z) = \prod(z_i - z_j) \qquad (3\ 29)$$

For correlators involving (3 27), one can integrate out the unitary matrix and off-diagonal T_{ij} In the remaining integration over z_i, the van der Monde determinant $\Delta(z)$ can be absorbed in holomorphic wavefunctions $\psi(z_1, \ldots z_N)$, leading to complete antisymmetrization This leads to a system of free fermions in a harmonic oscillator potential (the potential comes from the conformal coupling of the scalars to S^3)

The vacuum state corresponds to the filled Fermi sea

$$|F_0\rangle = \prod_{m=0}^{N-1}\psi_m^\dagger|0\rangle \to \mathrm{Det}\, z_i^{j-1}\exp\left[-\sum_{k=1}^{N}|z_k|^2\right] \qquad (3\ 30)$$

Here $\psi_m^\dagger$ is the creation operator for a fermion in the single-particle harmonic oscillator eigenstate $m = 0, 1, \ldots \infty$ The last expression is the wave-function in the holomorphic representation and is a Slater Determinant of the first N eigenstates The wavefunction also contains a factor depending on T_{ij}, $i > j$ which is not important to us

The operators (3 27) can be generated by multiplying elements of any of the following three sets of gauge invariant operators

- Symmetric Schur polynomials

$$S_n = \frac{1}{n!}\sum_{\sigma\in S_n} Z_{1,\sigma(1)} \ldots Z_{N,\sigma(N)} \qquad (3\ 31)$$

- Antisymmetric Schur polynomials

$$\tilde{\Sigma}_n = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma Z_{1,\sigma(1)} \ldots Z_{N,\sigma(N)} \qquad (3\ 32)$$

- Single Trace operators

$$O_n = \mathrm{Tr}Z^n \qquad (3\ 33)$$

It has been argued that these three correspond, in the bulk, to dual giant gravitons, giant gravitons and gravitons respectively (all at energy n) The picture of holes and particles arises as follows When one applies, say S_{sym}, (/) on the vacuum (3 30), one gets the Slater determinant of the first $N - 1$ states of the harmonic oscillator and an excited state with energy $N + n$. For multiple giant gravitons or dual giant gravitons, the discussion is more involved and the full solution is provided by exact bosonization of the fermions (see Section 3 4 2).

3.4.1. Interpretation of $u(x)$:

The AdS/CFT correspondence between (a) an LLM geometry, given by $u(x)$ and (b) a boundary state $|F\rangle$ (see (3 38)) is given by

$$u(x) = \mathrm{Tr}(\hat{U}(q,p)\rho)\Big|_{q=x_1, p=x_2}$$

$$\hat{U}(q,p) \equiv \int d\eta \psi^\dagger(x+\eta/2)\psi(x-\eta/2)\exp[ip\eta]$$

$$\rho = |F\rangle\langle F| \tag{3.34}$$

where $\psi(x) = \sum_n \chi_n(x)\psi_n$ is the Fermion field ($\chi_n(x) = H(x)\exp[-x^2/2]$) The operator $\hat{U}(p,q)$ is the Wigner distribution Other distributions such as Husimi can also be used in the semiclassical limit

3.4.2 *Bosonization* :

In [15] an exact bosonization of the N fermions in one dimension was found We will consider the general case first and go back to the fermions of the previous section later

General case :

Consider a system of fermions described by oscillators $\psi, \psi_n^\dagger$ $m, n = 0, 1, \ldots, \infty$ satisfying

$$\{\psi_m, \psi_n^\dagger\} = \delta_{mn} \tag{3.35}$$

We will consider two kinds of hamiltonians free

$$H_0 = \sum_m \mathcal{E}_m \psi_m^\dagger \psi_m, \tag{3.36}$$

and interacting :

$$H = H_0 + \sum_{m\,n\,r\,s} V_{m\,n,r,s}\psi_m^\dagger\psi_n\psi_r^\dagger\psi_s \tag{3.37}$$

We shall restrict to the N-particle Fock space, spanned by

$$|F\rangle = \prod_{m=1}^{N} \psi_{f_m}^\dagger |0\rangle, f\,N > \ldots > f_1 \geq 0, \psi_m|0\rangle = 0 \tag{3.38}$$

Only fermion ***bilinears*** of the form $\psi_n^\dagger\psi_m$, and their products, are now admissible operators In [15] two kinds of bosonic operators (denoted by a, b) were constructed out of these fermion bilinears (and *vice versa*)

1 Arbitrary number of bosons with a finite number of single-particle levels

$$a_i, a_j^\dagger, i, j = 1, \ldots, N, [a_i, a_j^\dagger] = \delta_{ij} \tag{3.39}$$

The states of the theory are given by

$$\prod_i^{l} (a_i^\dagger)^{r_i} |0\rangle, a_i|0\rangle = 0, r_i \geq 0 \tag{3.40}$$

The fermion state (3.38) corresponds to the following occupation numbers

$$r_N = f_1, r_k = f_{N-k+1} - f_{N-k} - 1, k = 1, 2, \ldots N-1 \tag{3.41}$$

The hamiltonian (3.36) gets mapped to

$$H_a = \sum_{k=1}^{N} \mathcal{E}\left(\sum_{i=N-k+1}^{N} a_i^\dagger a_i + k - 1\right) \tag{3.42}$$

This is, in general, *not free*. The interacting fermion hamiltonian (3.37) can in principle be bosonized by expressing the fermion bilinears in terms of the a, $a^\dagger$, but can be fairly complicated in practice (except for long range forces)

The states (3.40) approximately correspond to the multiple giant graviton states (3.16) constructed earlier from geometric quantization

2 N Bosons in an infinite number of single-particle levels

$$b_m, b_n^\dagger, m, n = 0, 1, \ldots, \infty, [b_n, b_n^\dagger] = \delta_{mn} \tag{3.43}$$

The states of the theory are given by

$$\prod^{N} b_{s_i}^\dagger |0\rangle, b_n|0\rangle = 0, 0 \leq s_N \leq \quad \tag{3.44}$$

The fermion state (3.38) maps to the following occupied levels

$$s_{N-i} \mapsto f_{i+1} - i, i = 0, \ldots, N-1 \tag{3.45}$$

The bosonic energy of this state is given by

$$E_b = \sum^{N} \mathcal{E}_{s_{N-i}+i} \tag{3.46}$$

The operator hamiltonian can be found in [15] and we will not write it explicitly here

The states (3.44) approximately correspond to the multiple dual giant graviton states states (3.15) constructed earlier from geometric quantization

Gravitons :

In [15] a third kind of boundary operator β_n, $\beta_n^\dagger$ was also discussed These were constructed from the fermion representation of the single trace operators $O_n = TrZ^n$. These *do not* satisfy the Heisenberg algebra, and for $n > N$ are given by the lower β's One can define multiple-beta states directly in terms of the fermion theory

$$\prod_n \beta_n^\dagger |F_0\rangle \tag{3.47}$$

These, for low energies, approximately correspond to the states (3.24)

3.4.3. *Bosonization of the half-BPS system* :

In this case the fermion system is free, $H = H_0$ (see (3.36)), with harmonic oscillator energy levels .

$$\mathcal{E}_m = m + 1/2 \tag{3.48}$$

Results

- By using (3 42) and (3 46) one can deduce that both the bosonic systems are also free and have harmonic oscillator spectrum The a-system has a uv cut-off $i \le N$ whereas the b-system does not
- The single-particle excitations $a_i^\dagger|0\rangle_a$ correspond to (single) giant gravitons ([15]) Similarly 'single'-particle excitations $b_n^\dagger|\bar{0}\rangle, |\bar{0}\rangle \equiv (b_0^\dagger)^{N-1}|0\rangle_b$ correspond to (single) dual giant gravitons See Section 3 5 for more
- Each of the above excitations corresponds to multi-graviton states, the latter being defined in terms of operators (3 27) Similarly each graviton is a multi-giant graviton state Thus, although there is no equivalence at the level of single particles, their Fock spaces are identical

$$\mathcal{F}_c = \mathcal{F}_{g} = \mathcal{F}_{dg} \tag{3.49}$$

- The gravitons cease to make sense as perturbative quanta for energies $\sim\sqrt{N}$ since their 3-pt correlators exponentially diverge with N The correlation function of the giant gravitons at large energies are however, well-defined and they continue to define sensible perturbative quanta In fact, the a and b bosons, which are closely related, provide a description in terms of free (!) quanta (this corresponds to an exactly diagonal basis of the hamiltonian)
- The bosonization resolves the puzzle of two giant-gravitons on top of each other (same with dual giants), which would seem to put two 'holes' (or two 'particles') on top of each other which violates Pauli exclusion It turns out that the distribution of giant gravitons is *not* the u of LLM, but some related distribution u_B which looks like an uneven pancake of radius r_0 (see footnote 2)

Table 1. Boundary operators

$3 = 3$	O_3	S_3	S_3'
$3 = 2+1$	O_2O_1	S_2S_1	$S_2'S_1'$
$3 = 1+1+1$	O_1^3	S_1^3	$S_1'^3$

3 5 Explicit counting of states and finite N

In the following we regard the gauge group in the boundary theory to be $U(N)$ We will explicitly count gauge-invariant operators for various values of the Hamiltonian H ($H = \Delta$, the scaling dimension in the Euclidean conformal field theory at the boundary) We will also count the bulk states in terms of gravitons/giant gravitons At each energy/dimension $\Delta = H = n$, we will find that the number (denoted by $g_N(n)$) is the same in the boundary and in the bulk and is given by

$$\mathcal{Z}(q,\zeta) = \prod_{n=1}^{\infty}(1-\zeta q^n)^{-1} = \sum_{M,n=0}^{\infty} \zeta^M q^n g_M'(n)$$

$$g_N(n) = \sum_{M=1}^{N} g_M'(n) \tag{3.50}$$

The assertion that $g_N(n)$ is given by (3 50) can be proved in a straightforward manner To see how the proof goes, let us evaluate the first few terms in $\mathcal{Z}(q, \zeta)$

$$\mathcal{Z}(q, \zeta) = 1 + \zeta q + (\zeta + \zeta^2)q^2 + (\zeta + \zeta^2 + \zeta^3)q^3 + \tag{3.51}$$

It is clear that the coefficient $g_M'(n)$ of $\zeta^M q^n$ counts the number of distinct ways in which the number n can be divided into M numbers Hence $g_N(n)$ denotes the total number of distinct partitions of the number n (up to $M = N$)

$$n = \nu_1 + \cdots \nu_l, l \le N, \nu_l \ge \ldots \ge \nu_1 > 0 \tag{3.52}$$

Thus, for $N = \infty$, the coefficient of q^3 in (3 51) corresponds to the following counting in the gauge theory

The first column corresponds to the partitions of the number 3 The second column lists the corresponding 'traces'(*cf* (3 27)), while the 3rd and 4th columns products of (3 31) and (3 32) respectively

The corresponding wavefunctions in the bulk are given by the table See (3 24), (3 15) and (3 16) Note the inverted order for the second column *vs.* the third column (this corresponds to the interchange of rows and columns between giants and dual giants)

3 5 1 Finite N

The expression (3 50) for $g_N(n)$ is valid for finite N The counting in the boundary theory reproduces this number So does the counting in terms of giant and dual giant gravitons Thus, if $N = 2$, $g_N(3) = g_2(3) = g_1'(3) + g_2'(3) = 2$ (see last line of (3 50); this corresponds to the deletion of the first row in the first column in Table 1) Correspondingly, in the same table the boundary operator in the first row of each column is a linear combination of those in the second and third rows, and should not be counted separately, indeed in the third column, corresponding to giant gravitons, $S_3' = 0$ identically, by definition of the antisymmetric Schur polynomials (3 32) In the second table (Table 2), similarly, the first row of the third column (corresponding to giant gravitons) vanishes identically since for $N = 2$ there is no excitation $\varphi_3^\dagger$! For

Table 2 Bulk excitations

	$c_3^\dagger\|0\rangle$	$(\varphi_1^\dagger)^3\|0\rangle$	$\varphi_3^\dagger\|0\rangle$
$3 = 2+1$	$c_2^\dagger c_1^\dagger\|0\rangle$	$\varphi_2^\dagger\varphi_1^\dagger\|0\rangle$	$\varphi_2^\dagger\varphi_1^\dagger\|0\rangle$
$3 = 1+1+1$	$(c_1^\dagger)^3\|0\rangle$	$\varphi_3^\dagger\|0\rangle$	$(\varphi_1^\dagger)^3\|0\rangle$

...e dual giant graviton (second column) the first row ...sappears because it involves three ($> 2 = N$) particles

... Boundary description of singular geometries .

...he superstar geometry has a naked singularity and ...responds to $u(x) = (1/1+Q) < 1$ in the LLM metric ...is can be obtained from (3 34) by choosing ρ to be a ...xed state [9]

4 1/8 BPS systems

4.1 Giant and dual giant gravitons

...e giant gravitons and dual giant gravitons corresponding ...1/8 supersymmetries have been analyzed in detail The ...alysis of the 1/8-th dual giants is particularly simple and ...es pretty much along the lines of their half-BPS cousins. These again wrap the 3-sphere of AdS_5, but they move on ...erent maximal circles of S^5, which are rotated with ...spect to each other by a $U(3) \subset O(6)$ rotation

The SUSY constraints can again be enforced by Dirac's ...thod The six coordinates of the generic dual giant are ...θ φ, χ_1, χ_2, χ_3, the last 5 angles parametrizing S^5 Because of SUSY constraints, this 6D space becomes a ...se space In fact the Dirac brackets imply that the ...ace is symplectically C^3 and the Hamiltonian in the ...uced phase space is that of a 3D SHO! The giant ...itons are more involved, but ultimately gives rise to ...e same spectrum There are indications that the two ...scriptions are dual to each other, however a proof of the ...ality in a manner similar to the half-BPS case described ...ove is still lacking

In the half-BPS case, the two kinds of giant gravitons ...ere not only dual to each other, they were, in a certain ...e, also dual to the gravitons (see (3.49)) In the 1/8 BPS case, the story of the gravitons has not been studied ...great detail Finding the counterpart of the LLM ...geometries is also an important outstanding problem

4.2 Quantization and comparison with the boundary theory

We describe the result for dual giants. The coordinate ...ace for a dual giant $R_+ \times S^5$, under the 1/8-th BPS ...nstraints, turns into a phase space which is symplectically isomorphic to C^3 The method is similar to that described above for dual giants. The wavefunctions are similar to (3 15) and are given by those of N bosons in a 3D SHO The partition function is given by

$$\mathcal{Z}(\zeta, q) = \prod_n (1 - \zeta q_1^{n_1} q_2^{n_2} q_3^{n_3})$$

$$= g'_M(J)\zeta^M q_1^{J_1} q_2^{J_2} q_3^{J_3}$$

$$g_N(J) = \sum_{M=0}^{N} g'_M(J) \tag{4 1}$$

The boundary operators (bosonic) are given by

$$\mathrm{Tr}(X^{n_1} Y^{n_2} Z^{n_3}) \tag{4 2}$$

and their products The counting of these operators agrees with (4 1), even at finite N Here (J_1, J_2, J_3) are angular momenta on the S^5

5. Less supersymmetries

5.1 Sasaki-Einstein geometries .

The various $AdS_m \times S^n$ backgrounds preserve 32 supersymmetries One may want to deform these geometries to obtain less supersymmetric backgrounds One way to achieve this, in case of $(m,n) = (4,7)$ or $(m,n) = (5,5)$ is to replace S^n by the so-called Sasaki-Einstein manifolds Y^n which reduces supersymmetry to a quarter Dual giant gravitons can again be considered in these geometries and quantized as above The coordinate space $R_t \times Y^n$ again becomes a phase space under BPS constraints and becomes symplectically isomorphic to the $(n+1)/2$ complex dimensional Kahler cone C over Y^n The Kahler quantization of these spaces, subject to a maximum number N of dual giant gravitons, appears to exactly reproduce the number of gauge-invariant operators of the $SU(N)$ sector of the boundary theory

5.2 $N_{susy} < 2$ ·

We will merely confine to quoting a few basic references since this will take us too far afield from the present discussion One of the main motivations of studying the BPS sector of these geometries is to understand the supersymmetric black holes that exist for two supersymmetries (1/16 of 32) Developments in this direction include [36] There has also been significant progress in understanding non-supersymmetric black holes and the corresponding boundary objects Some developments of interest in the present context are in [37,38]

6. Solution of Tomonaga's problem

We will not attempt to discuss technical details and will rather confine ourselves to a brief summary In [17] the tools recently developed in [15] are used to obtain an exact bosonization of a finite number N of non-relativistic fermions to solve the classic Tomonaga problem It is shown that the standard cubic effective hamiltonian for a massless relativistic boson arises in a systematic large-N and low-energy limit At finite N and high energies, however the low-energy effective description breaks down and the exact bosonized hamiltonian must be used A curious feature of this exact bosonized theory is that there is no underlying space visible The latter emerges only in the semiclassical (large-N) limit at low energies The bosonized theory of [17] thus provides an interesting example of emergent spacetime geometry which is expected to be a generic property of any consistent theory of quantum gravity

The solution to Tomonaga's problem has an interesting application to pure Yang-Mills theory on a cylinder which also reduces to free fermions on a circle In the context of the recent discussion of baby universes in string theory black holes, the results in [17] show that the O(N) piece in the bosonized hamiltonian written there provides a simple model for understanding the origin of two different kinds of nonperturbative O(e^{-N}) corrections to the partition function Note that in the discussion of bosonization of half-BPS sector of $AdS_5 \times S^5$, the bosonic oscillators of [15] turned out to create single-particle giant graviton states from the $AdS_5 \times S^5$ ground state It is worthwhile to figure out whether our bosonic oscillators also have a natural interpretation in the baby universe context

The bosonization of [17] appears to be generalizable also to higher space dimensions An interesting aspect of the bosonized theory seems to be the absence of a manifest reference to the number of dimensions This is not really surprising since space-time emerges only in the semiclassical low-energy limit in this bosonized theory It would be interesting to further explore the bosonized theory in higher dimensions In particular, it would be interesting to see how the bosonized theory encodes symmetries, *e g* spatial rotations

Acknowledgment

I would like to thank Aniket Basu, Avinash Dhar, Jaume Gomis, Shiraz Minwalla, Rob Myers, Nemani Suryanarayana and Spenta Wadia for discussions

References

[1] O Aharony, S S Gubser, J M Maldacena, H Ooguri and Y Oz *'Large N field theories, string theory and gravity,' Phys Rep* **323** 183 (2000) [arXiv hep-th/9905111]

[2] R Bousso *'The holographic principle,' Rev Mod Phys* **74** 825 (2002) [arXiv hep-th/0203101]

[3] H Lin, O Lunin and J Maldacena *'Bubbling AdS space and 1/2 BPS geometries', J High Energy Phys* **0410** 025 (2004) [hep-th/0409174]

[4] S Corley, A Jevicki and S Ramgoolam *Exact correlators of giant gravitons from dual N=4 SYM theory, Adv Theor Math Phys* **5** (2002) [hep-th/0111222]

[5] D Berenstein *A toy model for the AdS/CFT correspondence, J High Energy Phys* **0407** 018 (2004) [hep th/0403110]

[6] G Mandal *'Fermions from half BPS supergravity,' JHEP* **0508** 052 (2005) [arXiv hep-th/0502104]

[7] A Dhar *'Bosonization of non-relativistic fermions in 2-dimensions and collective field theory,' JHEP* **0507** 064 (2005) [arXiv hep-th/0505084]

[8] L Grant, L Maoz, J Marsano, K Papadodimas and V S Rychkov *'Minisuperspace quantization of 'bubbling AdS' and free fermion droplets,' JHEP* **0508** 025 (2005) [arXiv hep-th/0505079], L Maoz and V S Rychkov *'Geometry quantization from supergravity The case of 'bubbling AdS' JHEP* **0508** 096 (2005) [arXiv hep-th/0508059]

[9] V Balasubramanian, V Jejjala and J Simon *'The library of Babel' Int J Mod Phys D* **14** 2181 (2005) [arXiv hep th/0505123] V Balasubramanian, J de Boer, V Jejjala and J Simon *'The library of Babel On the origin of gravitational thermodynamics,' JHEP* **0512** 006 (2005) [arXiv hep-th/0508023]

[10] S Arapoglu, N S Deger, A Kaya, E Sezgin and P Sundell *'Multi spin giants,' Phys Rev D* **69** 106006 (2004) [arXiv hep-th/0312191]

[11] M M Caldarelli and P J Silva *"Multi giant graviton systems, SUSY breaking and CFT,' JHEP* **0402** 052 (2004) [arXiv hep-th/0401213]

[12] A Mikhailov *'Giant gravitons from holomorphic surfaces' JHEP* **0011** 027 (2000) [arXiv hep-th/0010206]

[13] N V Suryanarayana *'Half-BPS giants, free fermions and microstates of superstars,' JHEP* **0601** 082 (2006) [arXiv hep-th/0411145]

[14] A V Ryzhov *'Quarter BPS operators in N = 4 SYM,' JHEP* **0111** 046 (2001) [arXiv hep-th/0109064]

[15] A Dhar, G Mandal and N V Suryanarayana *'Exact operator bosonization of finite number of fermions in one space dimension,' JHEP* **0601** 118 (2006) [arXiv hep-th/0509164]

[16] A Dhar, G Mandal and M Smedback *'From gravitons to giants' JHEP* **0603** 031 (2006) [arXiv hep-th/0512312]

[17] A Dhar and G Mandal *'Bosonization of non-relativistic fermions on a circle Tomonaga's problem revisited,'* [arXiv hep-th/0603154]

[18] I Biswas, D Gaiotto, S Lahiri and S Minwalla *'Supersymmetric states of N = 4 Yang-Mills from giant gravitons,'* [arXiv hep-th/0606087]

[19] J Kinney, J M Maldacena, S Minwalla and S Raju *'An index for 4 dimensional super conformal theories,'* [arXiv hep-th/0510251]

[20] C Romelsberger *'An index to count chiral primaries in N = 1 d = 4 superconformal field theories,'* [arXiv hep-th/0510060]

[21] G Mandal and N V Suryanarayana *'Counting 1/8-BPS dual-giants,'* [arXiv hep-th/0606088]

[22] A Basu and G Mandal *'Dual giant gravitons in AdS(m) x Y**n (Sasaki-Einstein),'* [arXiv hep-th/0608093]

[23] S Benvenuti, B Feng, A Hanany and Y H He *'Counting BPS operators in gauge theories: Quivers, syzygies and plethystics,'* [arXiv hep-th/0608050]

[24] D Martelli and J Sparks *'Dual giant gravitons in Sasaki Einstein backgrounds,'* [arXiv hep-th/0608060]

[25] H J Kim, L J Romans and P Van Nieuwenhuizen *'The Mass Spectrum Of Chiral N=2 D = 10 Supergravity On S**5,' Phys Rev D* **32** 389 (1985)

[26] J McGreevy, L Susskind and N Toumbas, *'Invasion of the giant gravitons from anti-de Sitter space,' JHEP* **0006** 008 (2000) [arXiv hep-th/0003075]

[27] M T Grisaru, R C Myers and O Tafjord, *'SUSY and Goliath,' JHEP* **0008** 040 (2000) [arXiv hep-th/0008015]

[28] A Hashimoto, S Hirano and N Itzhaki *'Large branes in AdS and their field theory dual ' JHEP* **0008** 051 (2000) [arXiv hep-th/0008016]

[29] D Martelli and J Sparks *'Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals, Commun Math Phys* **262** 51 (2006) [arXiv hep-th/0411238]

[30] S Benvenuti, S Franco, A Hanany, D Martelli and J Sparks *'An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals,' JHEP* **0506** 064 (2005) [arXiv hep-th/0411264]

[31] S Tomonaga *'Remark on Bloch's method of sound waves applied to many fermion problem,' Progr Theor Phys* **5** 304 (1950)

[32] See, e g Johannes Voit *'One-Dimensional Fermi liquids',* [arXiv cond-mat/9510014]

[33] N M J Woodhouse *"Geometric Quantization,"* (New York, USA Clarendon) (1992) p307 (Oxford mathematical monographs)

[34] A Echeverria-Enriquez, M C Munoz-Lecanda, N Roman-Roy and C Victoria-Monge *'Mathematical foundations of geometric quantization,' Extracta Math* **13** 135 (1998) [arXiv math-ph/9904008]

[35] A Dhar, G Mandal and S R Wadia *'Nonrelativistic fermions, coadjoint orbits of W(infinity) and string field, Mod Phys Lett A* **7** 3129 (1992) [arXiv hep-th/9207011]

[36] J B Gutowski and H S Reall *'Supersymmetric AdS(5) black holes,' JHEP* **0402** 006 (2004) [arXiv hep-th/0401042], J B Gutowski and H S Reall *'General supersymmetric AdS(5) black holes,' JHEP* **0404** 048 (2004) [arXiv hep-th/0401129]

[37] O Aharony, J Marsano, S Minwalla, K Papadodimas and M Van Raamsdonk *'The Hagedorn / deconfinement phase transition in weakly coupled large N gauge theories,' Adv Theor Math Phys* **8** 603 (2004) [arXiv hep-th/0310285], O Aharony, S Minwalla and T Wiseman *'Plasma-balls in large N gauge theories and localized black holes,' Class Quant Grav* **23** 2171 (2006) [arXiv hep-th/0507219]

[38] L Alvarez-Gaume, P Basu, M Marino and S R Wadia *'Blackhole / string transition for the small Schwarzschild blackhole of AdS(5) x S**5 and critical unitary matrix models,'* [arXiv hep-th/0605041]